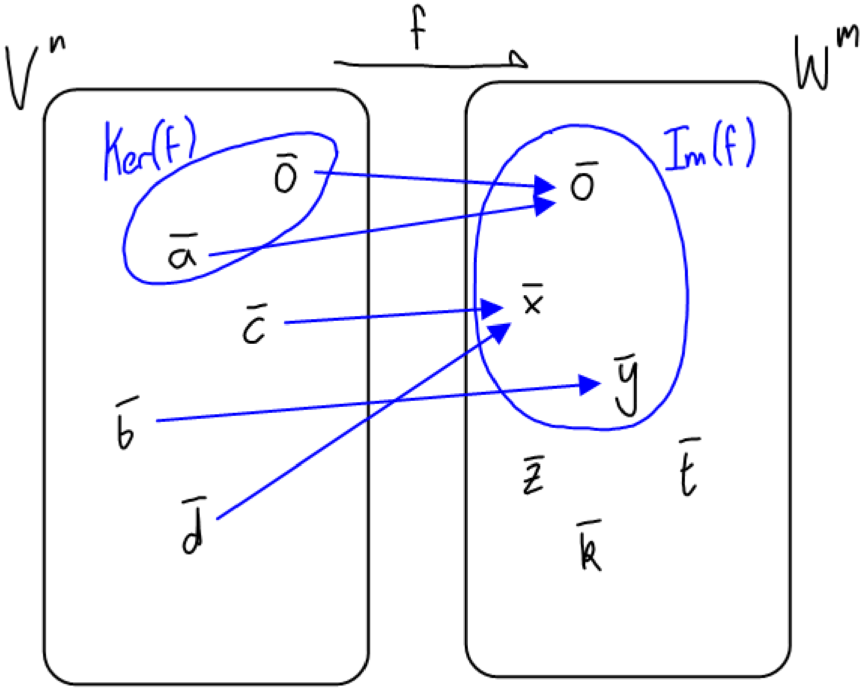
Homomorphisms in vector space

$$f : V^n \longrightarrow W^m$$
$$\bar{v} \in V^n \longrightarrow \bar{w} \in W^m$$
$$\Big\} f(\bar{v}) = \bar{w}$$

An application f will be called a homomorphism between two Vector Spaces V^n and W^m if: $\forall\, \bar{u}, \bar{v} \in V^n \quad f(\lambda \bar{u} + \mu \bar{v}) = \lambda f(\bar{u}) + \mu f(\bar{v})$



$$Ker(f) = \{ \bar{x} \in V \,/\, f(\bar{x}) = \bar{0} \in W \}$$

Ker(f) is a subspace of V

If $\dim(Ker(f)) = 0 \longrightarrow f$ is INYECTIVE

$$Im(f) = \{ \bar{y} \in W \,/\, \exists\, \bar{x} \in V \;\; f(\bar{x}) = \bar{y} \}$$

Im(f) is a subspace of W

$$Dim(Im(f)) = R_g(F_B)$$

Grassman's Law: dim(Ke...

The Matrix of a homomorfism

$f: V^n \longrightarrow W^m$ $\qquad$ $\forall \bar{x} \in V^n \quad f(\bar{x}) = \bar{y} \in W^m$

$B_1 = \{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ Base of V

$B_2 = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m\}$ Base of W

if $\bar{x} = x^1\bar{e}_1 + x^2\bar{e}_2 + \dots + x^n\bar{e}_n = (x^1, x^2, \dots, x^n)_{B_1}$

if $\bar{y} = y^1\bar{u}_1 + y^2\bar{u}_2 + \dots + y^m\bar{e}_m = (y^1, y^2, \dots y^m)_{B_2}$

$$\overbrace{\begin{pmatrix} \; & \; & \cdots & \; \end{pmatrix}}^{F_{B_1B_2}} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}_{B_1} = \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix}_{B_2} \Longleftrightarrow f(\bar{x}) = \bar{y} \in W^m$$

$$\underset{n \times m}{\left(F_{B_1B_2}\right)} \underset{n \times 1}{\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}} = \underset{m \times 1}{\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix}}$$

$f: V^3 \to V^3$ ENDOMORPHISM

$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ $\quad \forall \bar{x} \in V^3 \quad \bar{x} = x^1\bar{e}_1 + x^2\bar{e}_2 + x^3\bar{e}_3 = (x^1, x^2, x^3)_B$

If $f(\bar{x}) = \bar{y} \in V^3 \quad \bar{y} = y^1\bar{e}_1 + y^2\bar{e}_2 + y^3\bar{e}_3 = (y^1, y^2, y^3)_B$

$$\overbrace{\begin{pmatrix} f(\bar{e}_1) & f(\bar{e}_2) & f(\bar{e}_3) \end{pmatrix}}^{F_B} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_B = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_B$$

$f(\bar{e}_1)$ $f(\bar{e}_2)$ $f(\bar{e}_3)$ in B

3×3 $\quad 3\times1 \quad 3\times1$

Subspace operations

Given S_1, S_2 subspaces of V

$S_1 \cap S_2 = \{\bar{x} \in V \,/\, \bar{x} \in S_1 \overset{\text{AND}}{\wedge} \bar{x} \in S_2\} \longrightarrow S_1 \cap S_2$ is also a SUBSPACE

$S_1 \cup S_2 = \{\bar{x} \in V \,/\, \bar{x} \in S_1 \overset{\text{OR}}{\vee} \bar{x} \in S_2\} \longrightarrow S_1 \cup S_2$ is NOT a SUBSPACE

$S_1 + S_2 = \{\bar{x} \in V \,/\, \exists \bar{x}_1 \in S_1 \underset{\text{AND}}{\wedge} \exists \bar{x}_2 \in S_2 : \bar{x}_1 + \bar{x}_2 = \bar{x}\} \longrightarrow S_1 + S_2$ is also a SUBSPACE

Direct Sum : $S_1 \oplus S_2$

$$S_1 \oplus S_2 \iff \begin{cases} S_1 \cap S_2 = \{\bar{0}\} \\ \wedge \\ S_1 + S_2 = V \end{cases}$$

Vector spaces 1

Given $\mathbb{R}^3(\mathbb{R})$ with a Base $B=\{\bar{e}_1,\bar{e}_2,\bar{e}_3\}$ and an endomorphism f that works:

$f(x^1,x^2,x^3)=(x^1+x^2,\ x^1+x^2,\ x^3)$

1. Obtain $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ with a base and a dimension for each.
2. If that same space has two subspaces defined as: $S_1=\{\bar{x}\in\mathbb{R}^3 / x^1+x^2+x^3=0\}$ and $S_2=\{\bar{x}\in\mathbb{R}^3 / x^1-x^2=0\}$ obtain a base and a dimension for each as well as for $S_1\cap S_2$.
3. Given another base $B'=\{\bar{u}_1,\bar{u}_2,\bar{u}_3\}$ where $\bar{u}_1=\bar{e}_1+\bar{e}_2+\bar{e}_3$, $\bar{u}_2=\bar{e}_1+\bar{e}_2$ and $\bar{u}_3=\bar{e}_1$; obtain the calculated base of S_1 in this new base B'.

1. $B=\{\bar{e}_1,\bar{e}_2,\bar{e}_3\}$ $\quad f:\mathbb{R}^3\longrightarrow\mathbb{R}^3$

$f(\bar{x})=(x^1+x^2,\ x^1+x^2,\ x^3)$

$f(\bar{e}_1)=(1+0,1+0,0)=(1,1,0)_B$

$f(\bar{e}_2)=(0+1,0+1,0)=(1,1,0)_B$

$f(\bar{e}_3)=(0+0,0+0,1)=(0,0,1)_B$

$$F_B=\begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}$$

$(\bar{e}_3=(0,0,1)_B$

$$F_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \text{Number of variables (dimension of Space)} = \text{Number of equations} + \text{degrees of freedom (dimension of Subspace)} \end{array}}$$

Kernel

$$Ker(f) = \{ \bar{x} \in \mathbb{R}^3 / f(\bar{x}) = \bar{0} \}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{F_B} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_B \longrightarrow \begin{cases} x^1 + x^2 = 0 \\ \cancel{x^1 + x^2 = 0} \\ x^3 = 0 \end{cases} \longrightarrow \begin{cases} x^1 = \alpha \\ x^2 = -\alpha \quad \forall \alpha \in \mathbb{R} \\ x^3 = 0 \end{cases}$$

$$\dim(Ker(f)) = 1 \qquad B_K = \{(1, -1, 0)\}$$

$$\boxed{\begin{array}{l} \forall \bar{x} \in Ker \quad \bar{x} = (\alpha, -\alpha, 0) \quad \forall \alpha \in \mathbb{R} \\ \bar{x} = \alpha(1, -1, 0) \end{array}}$$

Image

$$Im(f) = \{ \bar{y} \in \mathbb{R}^3 / \bar{x} \in \mathbb{R}^3 \ f(\bar{x}) = \bar{y} \}$$

$$F_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B_{Im} = \{(1,1,0), (0,0,1)\}$$

$$\dim(Im(f)) = Ra(F_B) = 2$$

Alternate calculation of $Ker(f)$:

$$F_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$f(\bar{e}_1)$ $f(\bar{e}_2)$ $f(\bar{e}_3)$

$$f(\bar{e}_1) - f(\bar{e}_2) = \bar{0} \longrightarrow f(\bar{e}_1 - \bar{e}_2) = \bar{0} \longrightarrow \bar{e}_1 - \bar{e}_2 \in Ker(f) \longrightarrow B_K = \{(1,-1,0)\}$$

If f is a homomorph.:

$$f(\lambda\bar{u} + \mu\bar{v}) = \lambda f(\bar{u}) + \mu f(\bar{v})$$

$$Ker(f) = \{\bar{x} \in V \,/\, f(\bar{x}) = \bar{0} \in W\}$$

2.

$$S_1 = \{\bar{x} \in \mathbb{R}^3 / x^1 + x^2 + x^3 = 0\} \longrightarrow x^1 + x^2 + x^3 = 0 \longrightarrow \begin{cases} x^1 = \alpha \\ x^2 = \beta \\ x^3 = -\alpha - \beta \end{cases} \quad \forall \alpha, \beta \in \mathbb{R}$$

$\dim(S_1) = 2$

$B_{S_1} = \{(1,0,-1)_B, (0,1,-1)_B\}$

if $\bar{x} \in S_1$ $\quad \bar{x} = (\alpha, \beta, -\alpha-\beta) = \alpha(1,0,-1) + \beta(0,1,-1)$

$$\begin{cases} x^1 = \gamma \\ x^2 = \gamma \end{cases} \quad \forall \gamma, \delta \in \mathbb{R}$$

$\dim(S_2) = 2$

$B_{S_2} = \{(1,1,0),(0,0,1)\}$

$$S_1 \cap S_2 = \begin{cases} x^1 + x^2 + x^3 = 0 \\ x^1 - x^2 = 0 \end{cases}$$

$$\begin{cases} x^1 = \lambda \\ x^2 = \lambda \\ x^3 = -2\lambda \end{cases} \quad \forall \lambda \in \mathbb{R} \longrightarrow \begin{cases} \dim(S_1 \cap S_2) = 1 \\ B_{S_1 \cap S_2} = \{(1,1,-2)\} \end{cases}$$

3.

$$B_{S_1} = \{\overbrace{(1,0,-1)_B}^{\bar{a}_1}, \overbrace{(0,1,-1)_B}^{\bar{a}_2}\} \longrightarrow \begin{array}{l} \bar{a}_1 = (1,0,-1)_B = \bar{e}_1 - \bar{e}_3 \\ \bar{a}_2 = (0,1,-1)_B = \bar{e}_2 - \bar{e}_3 \end{array}$$

$$B \underset{C^{-1}}{\overset{C}{\rightleftarrows}} B'$$

$$B' \begin{cases} \bar{u}_1 = \bar{e}_1 + \bar{e}_2 + \bar{e}_3 = (1,1,1)_B \\ \bar{u}_2 = \bar{e}_1 + \bar{e}_2 = (1,1,0)_B \\ \bar{u}_3 = \bar{e}_1 = (1,0,0)_B \end{cases}$$

$$C = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccc} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{array} \\ \text{in } B \end{array}$$

$$|C| = -1$$

$$\mathrm{Adj}(C) = \begin{pmatrix} +0 & -0 & +-1 \\ -0 & +-1 & -1 \\ +-1 & -1 & +0 \end{pmatrix}$$

$$(C^{-1}) \overbrace{\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}_B}^{\bar{x}} = \overbrace{\begin{pmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \hat{x}^3 \end{pmatrix}_{B'}}^{\bar{x}}$$

$$C^{-1} = \frac{\mathrm{Adj}(C)^t}{|C|} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$C^{-1}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}_B = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}_{B'}$$

$$C^{-1}\begin{pmatrix} 0 \\ \cdots \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ \cdots \end{pmatrix}$$

$$\bar{a}_2 = (-1,2,-1)_{B'} = -\bar{u}_1 + 2\bar{u}_2 - \bar{u}_3$$

Vector Spaces 2

$\dim(\mathbb{P}_2) = 3$

$\{ \bar{x} \in \mathbb{R}^3 \, / \, \bar{x} = (a, b, c) \ \forall a, b, c \in \mathbb{R} \}$

Given $\mathbb{P}_2(\mathbb{R}) = \{ p(x) \in \mathbb{P}_2 \, / \, p(x) = ax^2 + bx + c \ \forall a, b, c \in \mathbb{R} \}$ and given an endomorphism with a matrix $F_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ in base $B = \{x^2, x, 1\}$; calculate $Ker(f)$ and $Im(f)$ giving a base of each and obtain $F_{B'}$, the matrix in base $B' = \{x^2+x+1, \ x^2+x, \ x^2\}$.

$Rg(F_B) = 1 = \dim(Im(f))$ $\qquad B_{Im} = \{ \underset{(1,1,1)_B}{x^2+x+1} \}$

$\underbrace{\dim(Im(f))}_{1} + \dim(Ker(f)) = \underbrace{\dim(\mathbb{P}_2)}_{3} \longrightarrow \dim(Ker(f)) = 2$

If f is a homomorph.:

$f(\lambda\bar{u} + \mu\bar{v}) = \lambda f(\bar{u}) + \mu f(\bar{v})$

$$F_B = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ f(x^2) \ f(x) \ f(1) \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{cases} a+b+c=0 \\ \cancel{a+b+c=0} \\ \cancel{a+b+c=0} \end{cases} \longrightarrow \begin{cases} a = \alpha \\ b = \beta \\ c = -\alpha - \beta \end{cases}$$

$\forall \alpha, \beta \in \mathbb{R}$

$f(x) - f(1) = \bar{0} \rightarrow f(x-1) = \bar{0} \longrightarrow x-1 \in Ker(f)$